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FUNCTIONAL ANALYTIC PROPERTIES OF TOPOLOGICAL SEMIGROUPS AND *N*-EXTREME AMENABILITY[1]

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Abstract. Let S be a topological semigroup, LUC(S) be the space of left uniformly continuous functions on S, and $\Delta(S)$ be the set of multiplicative means on LUC(S). If (*) LUC(S) has a left invariant mean in the convex hull of $\Delta(S)$, we associate with S a *unique* finite group G such that for any maximal proper closed left translation invariant ideal I in LUC(S), there exists a linear isometry mapping LUC$(G)/I$ one-one onto the set of bounded real functions on G. We also generalise some recent results of T. Mitchell and E. Granirer. In particular, we show that S satisfies (*) iff whenever S is a jointly continuous action on a compact hausdorff space X, there exists a nonempty finite subset F of X such that $sF=F$ for all $s\in S$. Furthermore, a discrete semigroup S satisfies (*) iff whenever $\{T_s; s\in S\}$ is an antirepresentation of S as linear maps from a norm linear space X into X with $\|T_s\|\leqq 1$ for all $s\in S$, there exists a finite subset $\sigma\subseteq S$ such that the distance (induced by the norm) of x from $K_X=$linear span of $\{x-T_sx;\, x\in X, s\in S\}$ in X coincides with distance of $O(\sigma, x)=\{(1/|\sigma|)\sum_{a\in\sigma} T_{at}(x);\, t\in S\}$ from 0 for all $x\in X$.

1. **Preliminaries and some notations.** Let S be a *topological semigroup* (i.e. a set with an associative multiplication and a hausdorff topology such that for each $a\in S$, the mappings $s\to a\cdot s$ and $s\to s\cdot a$, $s\in S$, are continuous from S into S) and X a hausdorff topological space. An *action* of S on X is a separately continuous mapping $S\times X\to X$ (i.e. continuous in each one of the variables when the other one of the variables is kept fixed) denoted by $(s, x)\to s\cdot x$, such that $s\cdot(t\cdot x)=(s\cdot t)\cdot x$ for all $s, t\in S$ and $x\in X$. An action on X is *jointly continuous* if the mapping $S\times X\to X$ is continuous when $S\times X$ has the product topology.

Let S be a topological semigroup which acts on a hausdorff topological space X, f be a bounded real function on X, ${}_sf(x)=f(sx)$ for all $s\in S$, $x\in X$ and $\|f\|=\sup_{x\in X}|f(x)|$; f is called *S-uniformly continuous* if f is continuous, and whenever $s_\alpha\to s_0$, $s_\alpha, s_0\in S$, then $\lim_\alpha \|{}_{s_\alpha}f-{}_{s_0}f\|=0$. We shall denote by

$m(X)=$ the space of bounded real functions on X,

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(1) A portion of the results in this paper is contained in the doctoral thesis of the author written under the direction of Professor E. E. Granirer at the University of British Columbia. The author is most indebted to Professor Granirer for his suggestions and encouragement.

$C(X)$ = the space of bounded continuous real functions on X,

LUC (S, X) = the space of S-uniformly continuous real functions on X.

When S is discrete, then $C(X) = \text{LUC}\,(S, X)$.

A subspace A of $m(X)$ is called *S-translation invariant* if ${}_sf \in A$ whenever $f \in A$ and $s \in S$. $C(X)$ [LUC (S, X)] is an S-translation invariant Banach algebra of $m(X)$ [$C(X)$] containing constants.

If T is a subset of X, then $1_T \in m(X)$ such that

$$1_T(t) = 1 \quad \text{if } t \in T, \qquad = 0 \quad \text{if } t \notin T.$$

Let A be a (norm) closed S-translation invariant subalgebra of $m(X)$ containing constants. For any $a \in S$, define $l_a\colon A \to A$ by $(l_a f)(x) = {}_af(x)$ for all $x \in X$ and $f \in A$ and $L_a\colon A^* \to A^*$ by $(L_a\phi)(f) = \phi({}_af)$ for all $f \in A$ and $\phi \in A^*$ (where A^* is the conjugate space of A). An element $\phi \in A^*$ is a *mean* if $\phi(f) \geqq 0$ for all $f \geqq 0$ and $\phi(1_X) = 1$; ϕ is *multiplicative* if $\phi(fg) = \phi(f)\phi(g)$ for all $f, g \in A$; and ϕ is *S-invariant* if $L_s\phi = \phi$ for all $s \in S$. The set of means on A is w^*- (i.e. $\sigma(A^*, A)$) compact.

For $a \in X$, let $p_a \in m(X)^*$ be the *point measure* at a, i.e. $p_a(f) = f(a)$ for all $f \in m(X)$. $\phi \in A^*$ is a point measure iff ϕ is the restriction to A of some point measure on $m(X)$. The set of point measure on A is w^*-dense in the set of multiplicative means on A [4, p. 275, proof of Corollary 19]. Furthermore, the set of multiplicative means on $m(X)$ is βX, the Stone-Čech compactification of X.

For any set A, $|A|$ will denote the *cardinality* of A. If A is a subset of a linear space, then Co A will denote the *convex hull* of A.

A topological semigroup S can be considered as an action on S defined by the mapping $(s, t) \to s \cdot t$, $s, t \in S$. In this case, the space of *left uniformly continuous* (i.e. S-uniform continuous) *functions* on S is denoted by LUC (S) (see Mitchell [16] and Namioka [17]).

For any topological semigroup S, and a positive integer n, LUC (S) is *left amenable* (LA) if LUC (S) has a *left invariant mean* (LIM), i.e. an S-invariant mean; LUC (S) is *n-extremely left amenable* (n-ELA), if there exists a subset H_0 of $\Delta(S)$, $|H_0| = n$, which is minimal with respect to the property $L_aH_0 = H_0$ for all $a \in S$, where $\Delta(S)$ denotes the set of multiplicative means on LUC (S). In this case, S/H_0 will denote the factor semigroup of S determined by the equivalence relation E: for any $a, b \in E$, $a\,E\,b$ iff $L_a\phi = L_b\phi$ for all $\phi \in H_0$. Note that if H_1 is another nonempty finite subset of $\Delta(S)$ which is minimal with respect to the property $L_aH_1 = H_1$ for all $a \in S$, then $|H_1| = n$. Furthermore, for any topological semigroup S, LUC (S) has a LIM in Co $\Delta(S)$ iff LUC (S) is n-ELA for some n [12, Remark 3.1 (a), (b)]. Namioka in [17] studies topological semigroups S for which LUC (S) is LA. Topological semigroups S for which LUC (S) is n-ELA have been studied by Mitchell in [16] (for $n = 1$) and the author in [12].

A semigroup S is LA [n-ELA] if S when considered as a discrete topological semigroup, LUC $(S) = m(S)$ is LA [n-ELA]. A recent survey of the theory of LA semigroups can be found in Day [3]. The class of ELA (i.e. 1-ELA) semigroups have been studied by Mitchell [14] and Granirer [6], [7] and [8]. Sorenson in [18] and the author in [12] consider the class of n-ELA semigroups for $n \geq 1$.

The class of ELA semigroups is immense (see for example, Granirer [6], [7], [8]) and n-ELA semigroups include product semigroups $S \times G$ where S is ELA and G is any group of order n. Furthermore, it has been shown by the author in [12] that for each n there exists a huge class of topological semigroups S for which LUC (S) is n-ELA and yet S is not even LA. However, if S is a subsemigroup of a locally compact group, then LUC (S) is n-ELA iff S is a finite group of order n (Granirer and Lau [9, Theorem 3]).

REMARK 1.1. Let S be a topological semigroup. The following are known and will be useful for our purpose:

(a) If LUC (S) has a LIM of the form $(1/n) \sum_1^n \phi_i$, where $\phi_i \in \Delta(S)$ (not necessarily distinct), then LUC (S) is m-ELA for some $m \leq n$, m divides n [12, Lemma 4.7].

(b) If LUC (S) is n-ELA, then there exists a collection $\mathscr{F}$ of n distinct open and closed subsets of S, with union S, such that $1_A \in$ LUC (S) for all $A \in \mathscr{F}$ and $\mathscr{F}$ is the decomposition of S by cosets of S/H for any finite subset $H \subseteq \Delta(S)$ satisfying $L_a H = H$ for all $a \in S$ [12, Theorem 4.1]. Consequently, S/H is a group of order n, and is a continuous homomorphic image of S. Furthermore, for any $\psi \in H$ and σ coset representative of S/H, $(1/n) \sum_{a \in \sigma} L_a \psi$ is a LIM on LUC (S).

2. **Actions of a topological semigroup on compacta.** The basis to our work lies on the generalisation of a fixed point theorem of T. Michell in [16, Theorem 1]. We first prove the following useful lemma:

LEMMA 2.1. *For any jointly continuous action of a topological semigroup S on a hausdorff topological space X, the mapping $S \to C(X)$ defined by $s \to {}_s f$, $s \in S$ and $f \in C(X)$ is continuous when $C(X)$ has the topology of uniform convergence on compacta. In particular, if X is compact, then* LUC $(S, X) = C(X)$.

Proof. Let $f \in C(X)$, $s_0 \in S$ and $\varepsilon > 0$. For any $x_0 \in X$, there exist neighbourhoods V_{x_0} of s_0 and U_{x_0} of x_0 such that

$$V_{x_0} \times U_{x_0} \subseteq \{(s, x);\ |f(s \cdot x) - f(s_0 \cdot x_0)| < \varepsilon\}.$$

If K is a compact subset of X, there exists $\{x_1, \ldots, x_n\} \subseteq K$ such that $\bigcup_{i=1}^n U_{x_i} \supseteq K$. Consequently, if $V = \bigcap_{i=1}^n V_{x_i}$, then

$$\sup_{x \in K} |{}_s f(x) - {}_{s_0} f(x)| < \varepsilon \quad \text{for all } s \in V.$$

For any topological semigroup S, by "S acts on $\Delta(S)$" we shall mean the jointly continuous mapping of $S \times \Delta(S) \to \Delta(S)$ defined by $(s, \phi) \to L_s \phi$ for all $\phi \in \Delta(S)$ and $s \in S$ (see Mitchell [16, proof of Theorem 1]).

THEOREM 2.2. *For any topological semigroup S and fixed n:*

(a) *If* LUC (S) *is n-*ELA, *then for any jointly continuous action of S on a compact hausdorff space X,*

(*) *there exists a nonempty subset $F \subseteq X$ such that $|F| \leqq n$, $|F|$ divides n and $aF=F$ for all $a \in S$.*

(b) *If S satisfies* (*) *when S acts on $\Delta(S)$, then* LUC (S) *is m-*ELA *for some $m \leqq n$, m divides n.*

Proof. (a) Let $H_0=\{\phi_1, \ldots, \phi_n\} \subseteq \Delta(S)$ be such that $L_a H_0 = H_0$ for all $a \in S$, and $z_0 \in X$ be fixed. Define $\psi_i \in$ LUC $(S, X)^*$ by $\psi_i(f)=\phi_i(T_{z_0} f)$ where $(T_{z_0} f)(s)=f(sz_0)$, $f \in$ LUC (S, X), $s \in S$ and $1 \leqq i \leqq n$. Then $\{\psi_i, 1 \leqq i \leqq n\}$ are multiplicative means on LUC (S, X) and $(1/n) \sum_1^n \psi_i$ is S-invariant on LUC $(S, X)=C(X)$ (Lemma 2.1). Since X is compact, each ψ_i becomes a point measure p_{x_i} on $C(X)$, $1 \leqq i \leqq n$, $x_i \in X$ [4, Lemma 2.5, p. 278]. Hence $\sum_1^n f(sx_i)=\sum_1^n f(x_i)$ for all $s \in S$, $f \in C(X)$. Since $C(X)$ separates closed sets, we have $sK=K$ for all $s \in S$, where $K=\{x_1, \cdots, x_n\}$. Furthermore, if $a, b \in S$ are such that $L_a\phi = L_b\phi$ for all $\phi \in H_0$, then $f(ax_i)=\phi_i(T_{z_0}({}_a f))=\phi_i(T_{z_0}({}_b f))=f(bx_i)$ for all $f \in C(X)$, $1 \leqq i \leqq n$. Since $C(X)$ separates points, we have $ax=bx$ for all $x \in K$. Consequently, we may consider the finite group $G=S/H_0$ of order n (Remark 1.1 (d)) as a group of transformation from K into K defined by $\bar{a}(x)=ax$ for all $x \in K$, $a \in S$, where $\bar{a}$ is the homomorphic image of a in S/H_0. Let $x_0 \in K$ be fixed and $F=\{gx_0; g \in G\}$. Then $gF=F$ for all $g \in G$. Define on G the equivalence relation E: for any $a, b \in G$, $a\,E\,b$ iff $ax_0=bx_0$. Let $\{g_1, \ldots, g_k\}$ be representatives from the equivalence classes $J_1, \ldots, J_k$ of G with respect to E, where $g_i \in J_i$ and g_1 is the identity of G. Since the mappings from H_1 onto H_i defined by $g \to g_i g$ for all $g \in H_1$ is one-one, $|J_1|=|J_i|$ for all $1 \leqq i \leqq k$. Consequently $|G|=k|J_1|$, i.e. $k=|F|$ divides $|G|=n$ and $sF=F$ for all $s \in S$.

(b) If $F \subseteq \Delta(S)$ is such that $L_a F=F$ for all $a \in S$, and $|F| \leqq n$, $|F|$ divides n, then $(1/|F|) \sum_{\phi \in F} \phi$ is a LIM on LUC (S). Our assertion now follows from Remark 1.1 (a).

REMARK 2.3. (a) When $n=1$, Theorem 2.2 reduces to Mitchell's fixed point theorem [16, Theorem 1].

(b) Note that in Theorem 2.2 (a), $F \subseteq X$ chosen in our proof satisfies the additional property that if F_0 is a nonempty subset of F such that $aF_0 \subseteq F_0$ for all $a \in S$, then $F_0=F$. For otherwise, let $g_0 x_0 \in F_0$, $g_0 \in G$ (x_0 and G are as in the proof), then $F=\{gg_0x_0; g \in G\} \subseteq F_0 \subseteq F$, i.e. $F_0=F$.

(c) In general $|F|$ in Theorem 2.2 (a) need not be n (e.g. consider the action of S on the discrete space $X=\{x_0\}$ defined by $sx_0=x_0$ for all $s \in S$). However, if n is prime, then F is either n or 1 (in this case, F becomes a fixed point).

A subsemigroup G of a semigroup S is a *left ideal group* if G is a group and a left ideal of S.

COROLLARY. *Let S be a compact topological semigroup with jointly continuous multiplication* (*i.e. the mapping from $S \times S \to S$ defined by $(s, t) \to s \cdot t$, $s, t \in S$ is* *continuous when* $S\times S$ *has the product topology*). *Then* LUC (S) *is* n-ELA *iff* S *has a left ideal group of* n *elements.*

Proof. If LUC (S) is n-ELA, then by applying Theorem 2.2 and Remark 2.3 (b) to the jointly continuous action of S on S defined by $(s, t)\to s\cdot t$, $s, t\in S$, we obtain a minimal left ideal F of S, $|F|\leqq n$. Necessarily F is a group. Since $H_0=\{p_s; s\in F\}\subseteq\Delta(S)$ satisfies $L_aH_0=H_0$ for all $a\in S$, $|F|=n$. The converse is trivial.

3. **Maximal translation invariant ideals.** In this section we associate with each topological semigroup S for which LUC (S) is n-ELA with a *unique* finite group G such that if I is a maximal proper closed left translation invariant (i.e. if $f\in I$ then ${}_sf\in I$ for all $s\in I$) ideal in LUC (S), then there exists a linear isometry mapping LUC $(S)/I$ one-one into $m(G)$.

THEOREM 3.1. *For any topological semigroup* S, LUC (S) *is* n-ELA *iff there exists a unique* (*up to isomorphism*) *finite group* G *which is a continuous homomorphic image of* S *such that* (*) G *has order* n *and for any maximal proper left translation invariant norm closed ideal* $I\subseteq$ LUC (S), *there exists a multiplicative linear isometry* T *mapping* LUC $(S)/I$ *one-one into* $m(G)$, *and* T *satisfies*:

(a) $T(\bar{1}_S)=1_G$,

(b) $T(\bar{f})\geqq 0$ *iff* $f\geqq 0, f\in$ LUC (S),

(c) $T({}_s\bar{f})={}_{\bar{s}}T(\bar{f})$ *for all* $s\in S, f\in$ LUC (S).

(*Note*: $\bar{s}$ *denotes the homomorphic image of* S *in* G *and* $\bar{f}$ *the equivalence class of* $f\in$ LUC (S) *in* LUC $(S)/I$.)

Proof. If LUC (S) is n-ELA, let G be the finite group (of order n) S/H_0, where H_0 is a finite subset of $\Delta(S)$ such that $L_aH_0=H_0$ for all $a\in S$. Then G is a continuous image of S (Remark 1.1 (b)). For any maximal proper left translation invariant closed ideal $I\subseteq$ LUC (S) (which exists by Zorn's lemma), let $\phi_0\in\Delta(S)$ be such that $\phi_0(f)=0$ for all $f\in I$ and $K=w^*$-closure of $\{L_s\phi_0; s\in S\}$. Applying Theorem 2.2 (a) to the jointly continuous action of S on K defined by $(s, \phi)\to L_s\phi$ for $s\in S$, $\phi\in K$, we obtain a finite subset $H_1\subseteq\Delta(S)$ such that $L_sH_1=H_1$ for all $s\in S$. Let $\psi_0\in H_1$ be fixed. If $a, b\in S$ and $f, g\in$ LUC (S) are such that $L_a\phi=L_b\phi$ for all $\phi\in H_0$ and $f-g\in I$, then $L_a\psi_0=L_b\psi_0$ (Remark 1.1 (b)) and hence $\psi_0({}_af-{}_bg)$ $=\psi_0({}_a(f-g))=0$. Consequently we may define a linear transformation T: LUC $(S)/I\to m(G)$ by $T(\bar{f})(\bar{s})=\psi_0({}_sf)$. It is simple to check that T is multiplicative and satisfies (a), (b) and (c). Furthermore, T is one-one since if $\bar{f}\neq\bar{g}$, then $f-g\notin I$ and hence $\psi_0({}_a(f-g))\neq 0$ for some $a\in S$, which implies $T(\bar{f})(\bar{a})\neq T(\bar{g})(\bar{a})$.

To see that T is onto, we let $a_1,\ldots, a_n$ be representative from the coset decomposition of S by $G=S/H_0$, $S_1,\ldots, S_n$ respectively. Then for each $1\leqq i\leqq n$, $1_{S_i}\in$ LUC (S) and $\mu=(1/n)\sum_1^n L_{a_i}\psi_0$ is a LIM on LUC (S) (Remark 1.1 (b)). Since $\mu(1_{S_1}+\cdots+1_{S_n})=1$, $S_1,\ldots, S_n$ are disjoint and $L_{a_i}\psi_0(1_{S_j})$ takes only values 0 or 1 for each i, j, it follows that for each i, $1 \leqq i \leqq n$, there exists exactly one element from $S_1, \ldots, S_n$, say S_i for convenience, such that $(L_{a_i}\psi_0)(1_{S_i})=1$. Now for any $\pi \in m(G)$, define $f=\sum_{i=1}^n \pi(\bar{a}_i)1_{S_i}$, then $T(\bar{f})=\pi$.

To show that T is an isometry, it suffices to show that for each $f \in \mathrm{LUC}(S)$, $\inf_{g\in I} \|f+g\| = \sup_{\phi\in K} |\phi(f)|$. In one direction $\sup_{\phi\in K} |\phi(f)| = \sup_{\phi\in K} |\phi(f+g)| \leqq \|f+g\|$ for all $g \in I$. In the other direction, we shall produce $g_0 \in I$ such that $\|f+g_0\| = \sup_{\phi\in K} |\phi(f)|$. We first observe that $\{f \in \mathrm{LUC}(S); \phi(f)=0 \text{ for all } \phi \in K\}$ is a closed left translation invariant proper ideal in LUC (S) containing I and hence equals to I. Let h be the continuous extension of f when restricted to K without increasing in norm, and put $g_0=h-f$. Then $g_0 \in I$ and $\|f+g_0\| = \|h\| = \sup_{\phi\in K} |\phi(f)|$.

G is unique for otherwise let $G'=\{t'_1, \ldots, t'_m\}$ be another finite group of order m which is a continuous homomorphic image of S satisfying (*), where $\bar{s}$, s' denote the homomorphic images of S in G, G' respectively. Let I be a maximal closed left translation invariant proper ideal in LUC (S); by assumption, there exists a multiplicative isometry T mapping LUC $(S)/I$ one-one onto $m(G')$ which satisfies conditions (a), (b) and (c). Define $\psi_i \in \Delta(S)$ by $\psi_i(f)=T(\bar{f})(t'_i)$ for each $1 \leqq i \leqq m$, $f \in \mathrm{LUC}(S)$. Let $H_2=\{\psi_1, \ldots, \psi_m\}$, then $L_s H_2 = H_2$ for all $s \in S$. For any $a, b \in S$, if $\bar{a}=\bar{b}$, then $L_a\psi = L_b\psi$ for all $\psi \in H_2$ (Remark 1.1 (b)) and hence $T(\bar{f})(a't'_i) = T(\bar{f})(b't'_i)$ for all $f \in \mathrm{LUC}(S)$, $1 \leqq i \leqq m$. Since T is onto and $m(G')$ separates points, we have $a't'_i = b't'_i$ for all $1 \leqq i \leqq m$, which implies $a'=b'$. Conversely if $a'=b'$, then $\psi_i({}_af) = T({}_a\bar{f})(t'_i) = T(\bar{f})(a't'_i) = T(\bar{f})(b't'_i) = \psi_i({}_bf)$ for all $1 \leqq i \leqq m$ and $f \in \mathrm{LUC}(S)$; hence $\bar{a}=\bar{b}$. Consequently, the mapping $G \to G_0$ defined by $\bar{a} \to a'$ is an isomorphism onto and hence $m=n$.

Conversely, if there exists a finite group G which satisfies (*), let T be a multiplicative linear isometry from LUC $(S)/I$ one-one onto $m(G)$ satisfying (a), (b) and (c), where I is a maximal closed left translation invariant proper ideal in LUC (S), and $G=\{\bar{s}_1, \ldots, \bar{s}_n\}$. Define $\phi_i \in \Delta(S)$ by $\phi_i(f)=T(\bar{f})(\bar{s}_i)$. Then $(1/n)\sum_1^n \varphi_i$ is a LIM on LUC (S). Consequently, LUC (S) is m-ELA for some $m \leqq n$ (Remark 1.1 (a)). By uniqueness of G and what we have proved, $m=n$.

4. n-ELA **semigroups.** Let n be fixed, X be a norm linear space, $\{T_s; s \in S\}$ be an antirepresentation of S as linear transformations from X into X such that $\|T_s\| \leqq 1$ for all $s \in S$, and $\sigma=\{a_1, \ldots, a_n\} \subseteq S$. We may define a linear transformation T_σ from X into X by $T_\sigma(x)=(1/n)\sum_1^n T_{a_i}(x)$ for all $x \in S$. Furthermore, if $x \in X$, denote by

$O(x)=\{T_s(x); s \in S\}$,

$O(\sigma, x)=\{T_{\sigma\circ t}(x); t \in S\}$,

$K_X=$ linear span of $\{x-T_sx; s \in S, x \in X\}$ in X.

Our next result, which partially generalises Theorem 5 I(b), II(b) of Granirer [8] asserts that if S is n-ELA, then $d(K_X, x)=d(O(\sigma, x), 0)$ for some finite subset $\sigma \subseteq S$, $|\sigma| \leqq n$ and all $x \in X$ and d is the metric induced by the norm on X. We first prove the following lemma:

LEMMA 4.1. *Let S be a semigroup with f.i.p.r.i. and $\sigma_0 \subseteq S$, $|\sigma_0| = n$. If for each $a \in S$, and $f \in m(S)$, there exists a net $\{t_\alpha\}$ in S, depending on a and f such that* $\lim \|l_{t_\alpha}(l_{\sigma_0}(f - l_a))\| = 0$, *then S is m-ELA for some $m \leqq n$, m divides n.*

Proof. For each $a \in S$ and $f \in m(S)$, let

$$K_{a,f} = \left\{\phi \in \beta S; \frac{1}{n}\sum_{s\in\sigma_0} L_s\phi({}_af) = \frac{1}{n}\sum_{s\in\sigma_0} L_s\phi(f)\right\}.$$

Then $K_{a,f}$ are nonempty, w^*-closed subsets of βS. Furthermore, the family $\{K_{a,f};\ a \in S, f \in m(S)\}$ has finite intersection property. In fact if $\alpha > 0$, $\{a_1, \ldots, a_k\} \subseteq S$ and $\{f_1, \ldots, f_k\} \subseteq m(S)$, let $t_1^\alpha, \ldots, t_k^\alpha$ be such that $\|l_{t_i^\alpha}(l_{\sigma_0}(f_i - {}_{a_i}f_i))\| < \alpha$. Let $c_\alpha \in \bigcap_{i=1}^k t_i S$, then $c_\alpha = t_i^\alpha s_i^\alpha$ for some $s_i^\alpha \in S$, and $\|p_{c_\alpha}(l_{\sigma_0}(f_i - {}_{a_i}f_i))\| \leqq \|l_{t_i^\alpha}(l_{\sigma_0}(f_i - {}_{a_i}f_i))\| < \alpha$ for $1 \leqq i \leqq k$. If ϕ is a cluster point of the net $\{p_{c_\alpha};\ \alpha > 0\}$, then $\phi \in \bigcap_{i=1}^k K_{a_i,f_i}$. By w^*-compactness of βS, there exists $\phi_0 \in \beta S$ such that

$$\frac{1}{n}\sum_{s\in\sigma_0} L_s\phi_0({}_af) = \frac{1}{n}\sum_{a\in\sigma_0} L_s\phi_0(f)$$

for all $f \in m(S)$, $a \in S$. By Remark 1.1 (a), S is m-ELA for some $m \leqq n$, m divides n.

Let S be a (discrete) semigroup with f.i.p.r.i. (i.e. finite intersection property for right ideals), define as in Granirer [5] on S the equivalence relation (r): $a\,(r)\,b$ iff $ac = bc$ for some $c \in S$. Then (r) is two-sided stable (i.e. $a\,(r)\,b$ implies $ac\,(r)\,bc$ and $ca\,(r)\,cb$ for all $a, b, c \in S$) and the factor semigroup induced by (r) is denoted by $S/(r)$.

REMARK 4.2. We will need in what follows the following known results:

(a) A semigroup S is n-ELA iff S has f.i.p.r.i. and $S/(r)$ is a group of order n (see Sorenson [18, Theorem 3.3.6] and Lau [12, Theorem 5.2]).

(b) If S is an n-ELA semigroup, and F_0 is a coset representative of $S/(\mathrm{r})$, then for each finite subset $\sigma \subseteq S$, there exists $t_\sigma \in S$, depending on σ, such that $aF_0t_\sigma = F_0t_\sigma$ for all $a \in \sigma$ (see Granirer [6, Theorem 1] for $n = 1$, and Lau [12, Theorem 5.3]).

THEOREM 4.3. (a) *Let X be a norm linear space and $\{T_s;\ s \in S\}$ be an antirepresentation of a semigroup S as linear transformations from X into X with $\|T_s\| \leqq 1$ for all $s \in S$. If S is n-ELA, then for each σ_0, coset representative of $S/(r)$,*

(*) $d(K_X, x) = d(O(\sigma_0, x), 0)$ *for all $x \in X$ where d is the metric induced by the norm on X.*

(b) *If S is a semigroup whose antirepresentation $\{l_s;\ s \in S\}$ on $m(S)$ (with* sup norm*) satisfies (*) for some $\sigma_0 \subseteq S$, $|\sigma_0| = n$, then S is m-ELA for some $m \leqq n$, m divides n.*

Proof. (a) Let $\|\cdot\|$ denote the norm on X and $x \in X$ be arbitrary but fixed. If $x_0 \in K_X$, then $x_0 = \sum_1^m k_i(x_i - T_{s_i}x_i)$ where k_i are scalars, $x_i \in X$, and $s_i \in S$, $i = 1, \ldots, m$. Choose $t_0 \in S$ such that $s_i\sigma_0t_0 = \sigma_0t_0$ for $i = 1, \ldots, m$. Then as readily checked, $T_{\sigma_0t_0}(x_0) = 0$ and

$$\|x - x_0\| \geqq \|T_{\sigma_0t_0}(x - x_0)\| = \|T_{\sigma_0t_0}(x)\|,$$

i.e. $d(K_X, x) \geqq d(O(\sigma_0, x), 0)$. Since

$$T_{\sigma_0 t}(x) = (1/n) \sum_{s \in \sigma_0} (T_{st}(x) - x) - x \quad \text{and} \quad (1/n) \sum_{s \in \sigma_0} (T_{st}(x) - x) \in K_X$$

for all $t \in S$, it follows that (*) holds.

(b) Condition (*) implies that $d(K_{m(S)}, 1) \geqq 1$ and hence $K_{m(S)}$ is not uniformly dense in $m(S)$. Consequently, S is left amenable (this follows from Proposition 3.2 of Namioka [17]; see also Day [3, Lemma 2.2]) and so S has f.i.p.r.i. Furthermore, if $f \in m(S)$, then $d(\text{Co } O(f), 0) = (1/n) d(O(\sigma_0, f), 0)$, where Co $O(f)$ is the convex hull of $\{l_s; s \in S\}$. In fact, $d(f + K_{m(S)}, 0) \geqq (1/n) d(O(\alpha_0, f), 0) \geqq d(\text{Co } O(f), 0) \geqq d(f + K_{m(S)}, 0)$. The last inequality follows from Co $O(f) \subseteq f + K_{m(S)}$ as readily checked and shown in [8, p. 59]. If $f \in m(S)$, $a \in S$, $\|(1/n) \sum_{i=1}^{n} (l_{a^i}(f - {}_a f)\| \leqq 2\|f\|/n \to 0$ if n is large; hence $d(O(\sigma_0, f - {}_a f), 0) = d(\text{Co } O(f - {}_a f), 0) = 0$ for all $a \in S$, and $f \in m(S)$. By Lemma 4.1 S is m-ELA for some $m \leqq n$, m divides n.

Remark. It follows from the proof of Theorem 4.3 that in order for S to be m-ELA for some $m \leqq n$, m divides n, it is sufficient for S to have f.i.p.r.i. and there exists $\sigma_0 \subseteq S$, $|\sigma_0| = n$ such that for each $f \in m(S)$, $d(\text{Co } O(f), 0) \geqq (1/n) d(O(\sigma_0, f), 0)$. For $n = 1$, as shown by Granirer [8, Theorem 5 II(b)], in order for S to be ELA, it is enough that for each $f \in m(S)$, $d(\text{Co } O(f), 0) = d(O(f), 0)$. However, we do not know whether or not in order for S to be m-ELA for some $m \leqq n$, $n \geqq 2$, it is enough to have $\sigma_0 \subseteq S$, $|\sigma_0| = n$ such that $d(\text{Co } O(f), 0) \geqq (1/n) d(O(\sigma_0, f), 0)$ for all $f \in m(S)$ without imposing the condition that S has f.i.p.r.i.

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